

Noncommutative Field Theory: Nonrelativistic Fermionic Field Coupled to the Chern-Simons Field in 2+1 Dimensions

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Abstract

We study a noncommutative nonrelativistic fermionic field theory in 2+1 dimensions coupled to the Chern-Simons field. We perform a perturbative analysis of model and show that up to one loop the ultraviolet divergences are canceled and the infrared divergences are eliminated by the noncommutative Pauli term.

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I. INTRODUCTION

The noncommutative Aharonov-Bohm (AB) effect for scalar particles has been studied in the quantum mechanical [1, 2] and in the field theory contexts [3]. As in the commutative case, in the latter case the effect was simulated by a nonrelativistic field theory of spin zero particles interacting through a Chern-Simons (CS) field. Differently from its commutative counterpart, however, the model turns out to be renormalizable even without a quartic self-interaction of the scalar field (the quartic self-interaction is however necessary if a smooth commutative limit is required). It is known, that in the commutative situation, the Pauli term plays for the spin 1/2 AB scattering [4] the same role as the quartic interaction plays for the case of scalar particles [5]. One may then conjecture that a noncommutative Pauli interaction is also not necessary at least as a pre-requisite for renormalizability. In this brief note we will prove that this conjecture indeed holds, the Pauli term being necessary to obtain a smooth result in the commutative limit but not to fix the ultraviolet renormalizability of the model. Our analysis is based in the (2+1) dimensional model described by the action

$$S[A, \psi] = \int d^3x \left[\frac{\kappa}{2} \varepsilon^{\mu\nu\lambda} \left(A_\mu \partial_\nu A_\lambda + \frac{2ig}{3} A_\mu * A_\nu * A_\lambda \right) - \frac{1}{2\xi} \partial_i A^i \partial_j A^j + i\psi^\dagger * D_t \psi - \frac{1}{2m} (\mathbf{D}\psi)^\dagger * (\mathbf{D}\psi) + \lambda \psi^\dagger * B * \psi + \partial^i \bar{c} \partial_i c + ig \partial^i \bar{c} * [A_i, c]_* \right], \quad (1.1)$$

where $B = -F_{12} = \partial_1 A^2 - \partial_2 A^1 + ig[A^1, A^2]$ and ψ is a one-component fermion field transforming in accord with the fundamental representation of the noncommutative $U(1)$ group (other noncommutative aspects of nonrelativistic fermions interacting with the CS field were considered in [6, 7])

$$\psi \rightarrow (e^{i\Lambda})_* * \psi, \quad (1.2)$$

$$\psi^\dagger \rightarrow \psi^\dagger * (e^{-i\Lambda})_*, \quad (1.3)$$

The covariant derivatives in Eq. (1.1) are given by

$$\begin{aligned} D_t \psi &= \partial_t \psi + ig A_0 * \psi, \\ D_i \psi &= \partial_i \psi + ig A_i * \psi. \end{aligned} \quad (1.4)$$

For convenience, we will work in a strict Coulomb gauge obtained by letting $\xi \rightarrow 0$. We will use a graphical notation where the CS field, the matter field and the ghost field propagators are represented by wavy, continuous and dashed line respectively. The graphical representation for the Feynman rules is given in Fig. 1 and the corresponding analytical expression are:

(i) The matter field propagator:

$$S(p) = \frac{i}{p_0 - \frac{\mathbf{p}^2}{2m} + i\epsilon}, \quad (1.5)$$

(iii) The ghost field propagator:

$$G(p) = -\frac{i}{\mathbf{p}^2}, \quad (1.6)$$

(iii) The mixed propagator:

$$\Delta_{A_0 B}(p) = -\frac{i}{\kappa}, \quad (1.7)$$

(iv) The gauge field propagator in the limit $\xi \rightarrow 0$ is

$$D_{\mu\nu}(k) = \frac{\varepsilon_{\mu\nu\lambda} \bar{k}^\lambda}{\kappa \mathbf{k}^2}, \quad (1.8)$$

where $\bar{k}^\lambda = (0, \mathbf{k})$.

(v) The analytical expressions associated to the vertices are:

$$\Gamma^0(p, p') = -ig e^{ip\theta p'}, \quad (1.9)$$

$$\Gamma^i(p, p') = \frac{ig}{2m} (p + p')^i e^{ip\theta p'}, \quad (1.10)$$

$$\Gamma_{ghost}^i(p, p') = -2gp^i \sin(p\theta p'), \quad (1.11)$$

$$\Gamma^{\mu\nu\lambda}(k_1, k_2) = 2ig\kappa \varepsilon^{\mu\nu\lambda} \sin(k_1\theta k_2), \quad (1.12)$$

$$\Gamma_B^{ij}(k_1, k_2, p, p') = 2ig\lambda \sin(k_1\theta k_2) e^{ip\theta p'} \varepsilon^{ij}, \quad (1.13)$$

$$\Gamma^{ij}(k_1, k_2, p, p') = -\frac{ig^2}{m} \cos(k_1\theta k_2) e^{ip\theta p'} \delta^{ij}, \quad (1.14)$$

$$\Gamma^B(p, p') = i\lambda e^{ip\theta p'}. \quad (1.15)$$

In these expressions we have defined $k_1\theta k_2 \equiv \frac{1}{2}\theta^{\mu\nu} k_{1\mu} k_{2\nu}$, where $\theta^{\mu\nu}$ is the antisymmetric matrix which characterizes the noncommutativity of the underlying space. For simplicity and to evade possible unitarity/causality problems [8] we keep time local by imposing $\theta^{0i} = 0$. We set also $\theta^{ij} = \theta \varepsilon^{ij}$ with ε^{ij} being the two dimensional Levi-Civita symbol, normalized as $\varepsilon^{12} = 1$.

II. PARTICLE-PARTICLE SCATTERING

A. Tree Level Scattering

In the tree approximation and in the center-of-mass frame the two body scattering amplitude, depicted in Fig. 2a, is given by

$$\mathcal{A}_a^0(\varphi) = -\frac{2ig^2(\mathbf{p}_1 \wedge \mathbf{p}_3)}{m\kappa} \left[\frac{e^{i(p_1\theta p_3 + p_2\theta p_4)}}{(\mathbf{p}_1 - \mathbf{p}_3)^2} - \frac{e^{-i(p_1\theta p_3 + p_2\theta p_4)}}{(\mathbf{p}_1 + \mathbf{p}_3)^2} \right], \quad (2.1)$$

where \mathbf{p}_1 , \mathbf{p}_2 and \mathbf{p}_3 , \mathbf{p}_4 are, respectively, the incoming and outgoing momenta. Since $\theta_{ij} = \theta\epsilon_{ij}$, the phase is $p_1\theta p_3 + p_2\theta p_4 = \theta(\mathbf{p}_1 \wedge \mathbf{p}_3) = \theta\mathbf{p}^2 \sin \varphi = \bar{\theta} \sin \varphi$, where we have defined $\bar{\theta} \equiv \theta\mathbf{p}^2$ and φ is the scattering angle. Therefore, Eq. (2.1) can be rewritten as

$$\mathcal{A}_a^0(\varphi) = -\frac{ig^2}{m\kappa} \left[\frac{e^{i\bar{\theta} \sin \varphi}}{1 - \cos \varphi} - \frac{e^{-i\bar{\theta} \sin \varphi}}{1 + \cos \varphi} \right] \sin \varphi. \quad (2.2)$$

For the graph in Fig. 2b, we have

$$\mathcal{A}_b^0(\varphi) = -\frac{4g\lambda}{\kappa} \cos(\bar{\theta} \sin \varphi). \quad (2.3)$$

Thus, the full tree level amplitude is

$$\mathcal{A}^0(\varphi) = -\frac{ig^2}{m\kappa} [\cot(\varphi/2)e^{i\bar{\theta} \sin \varphi} - \tan(\varphi/2)e^{-i\bar{\theta} \sin \varphi}] - \frac{4g\lambda}{\kappa} \cos(\bar{\theta} \sin \varphi), \quad (2.4)$$

furnishing up to first order in the parameter $\bar{\theta}$,

$$\mathcal{A}^0(\varphi) = -\frac{2ig^2}{m\kappa} (\cot \varphi + i\bar{\theta}) - \frac{4g\lambda}{\kappa} + \mathcal{O}(\bar{\theta}^2). \quad (2.5)$$

Notice that the noncommutative contribution is isotropic although energy dependent.

B. One Loop Scattering

The one-loop contribution to the scattering amplitude is depicted in Fig. 3. Two other diagrams corresponding to graphs 3b and 3c with the upper and bottom fermionic lines exchanged not explicitly shown. All other possible one-loop graphs vanish. The

expressions for the contributions of the box, triangle and trigluon graphs, shown in Figs. 3(a-c), are the same as in the scalar case [3] so that we just quote the results:

$$\mathcal{A}_a(\varphi) = -\frac{g^4}{2\pi m\kappa^2} [\ln(2\sin\varphi) + i\pi] - \frac{i\bar{\theta}g^4\sin\varphi}{\pi m\kappa^2} \ln\left[\tan\left(\frac{\varphi}{2}\right)\right] + \mathcal{O}(\bar{\theta}^2), \quad (2.6)$$

for the total contribution of the box graph,

$$\begin{aligned} \mathcal{A}_b^{np}(\varphi) &= \frac{g^4}{2\pi m\kappa^2} [\ln(\bar{\theta}/2) + \gamma] + \frac{g^4}{2\pi m\kappa^2} \ln(2\sin\varphi) \\ &\quad + \frac{i\bar{\theta}\sin\varphi g^4}{2\pi m\kappa^2} \ln[\tan(\varphi/2)] + \mathcal{O}(\bar{\theta}^2) \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} \mathcal{A}_b^p(\varphi) &= -\frac{g^4}{4\pi m\kappa^2} [\cos(\bar{\theta}\sin\varphi) \ln\left(\frac{\Lambda^2}{\mathbf{p}^2}\right) \\ &\quad - \ln|2\sin(\varphi/2)|e^{i\bar{\theta}\sin\varphi} - \ln|2\cos(\varphi/2)|e^{-i\bar{\theta}\sin\varphi}], \end{aligned} \quad (2.8)$$

for the nonplanar and planar parts of the triangle graph,

$$\begin{aligned} \mathcal{A}_c^{np}(\varphi) &= \frac{3g^4}{2\pi m\kappa^2} [\ln(\bar{\theta}/2) + \gamma] + \frac{3g^4}{2\pi m\kappa^2} \ln(2\sin\varphi) \\ &\quad + \frac{3i\bar{\theta}\sin\varphi g^4}{2\pi m\kappa^2} \ln[\tan(\varphi/2)] + \frac{2g^4}{\pi m\kappa^2} + \mathcal{O}(\bar{\theta}^2) \end{aligned} \quad (2.9)$$

and

$$\begin{aligned} \mathcal{A}_c^p(\varphi) &= \frac{g^4}{4\pi m\kappa^2} [\cos(\bar{\theta}\sin\varphi) \left[\ln\left(\frac{\Lambda^2}{\mathbf{p}^2}\right) + 1 \right] \\ &\quad - \ln|2\sin(\varphi/2)|e^{i\bar{\theta}\sin\varphi} - \ln|2\cos(\varphi/2)|e^{-i\bar{\theta}\sin\varphi}], \end{aligned} \quad (2.10)$$

for the nonplanar and planar parts of the trigluon graph.

The graphs containing the noncommutative Pauli vertex are depicted in Fig. 3d and Fig. 4(a,b).

The contribution of the graph in Fig. 3d, which is purely nonplanar, is given by

$$\mathcal{A}_d^{np}(\varphi) = \frac{4mg^2\lambda^2}{\kappa^2} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \left[\frac{e^{2iq\theta k} + e^{2iq'\theta k}}{(\mathbf{k}^2 - \mathbf{p}^2 - i\epsilon)} \right] \quad (2.11)$$

thus, by evaluating the integral in the momenta [9] gives

$$\mathcal{A}_d^{np}(\varphi) = -\frac{4mg^2\lambda^2}{\pi\kappa^2} \left[\ln\left(\frac{\bar{\theta}}{2}\right) + \gamma \right] - \frac{2mg^2\lambda^2}{\pi\kappa^2} \ln[2\sin\varphi] + \frac{2img^2\lambda^2}{\kappa^2} + \mathcal{O}(\bar{\theta}^2), \quad (2.12)$$

for small θ . As can be easily verified, the contributions from the other two graphs, shown in Figs. 4a and 4b cancel among themselves.

Summing all the contributions, we get the total one-loop amplitude

$$\begin{aligned}
\mathcal{A}_{1\text{-loop}}(\varphi) &= \mathcal{A}_{1\text{-loop}}^p(\varphi) + \mathcal{A}_{1\text{-loop}}^{np}(\varphi) + \mathcal{A}_a(\varphi) \\
&= -\frac{2ig^2}{m\kappa} \cot \varphi - \frac{4g\lambda}{\kappa} + \frac{2\bar{\theta}g^2}{m\kappa} + \frac{9g^4}{4\pi m\kappa^2} - \frac{ig^4}{2m\kappa^2} + \frac{2img^2\lambda^2}{\kappa^2} \\
&\quad + \left(\frac{3g^4}{2\pi m\kappa^2} - \frac{2mg^2\lambda^2}{\pi\kappa^2} \right) \ln[2 \sin \varphi] + \frac{i\bar{\theta}g^4 \sin \varphi}{\pi m\kappa^2} \ln[\tan(\varphi/2)] \\
&\quad + \frac{4mg^2}{\pi\kappa^2} \left(\frac{g^2}{2m^2} - \lambda^2 \right) [\ln(\bar{\theta}/2) + \gamma] + \mathcal{O}(\bar{\theta}^2). \tag{2.13}
\end{aligned}$$

For $\lambda = \pm \frac{g}{\sqrt{2m}}$, the limit $\bar{\theta} \rightarrow 0$ is analytical. Hence, we show that the scattering amplitude up to one-loop order for the noncommutative nonrelativistic fermionic field theory in 2+1 dimensions coupled to the Chern-Simons field does not present ultraviolet divergences and the IR divergences in the scattering amplitude are canceled by the noncommutative Pauli term.

In this work we have studied the two body scattering amplitude when the colliding particles both have either spin up or spin down. If the particles in colliding beams have opposite spins the contribution of the noncommutative Pauli terms cancels. In this case, designating by ψ and ϕ the fermionic fields associated with the particles in the beams, to get a smooth commutative limit it will be necessary to include a quartic term $\phi^\dagger * \phi * \psi^\dagger * \psi$. In fact, in the commutative situation one such term is induced if one starts from the relativistic theory [10] and integrates over the high energy modes to get an effective nonrelativistic field theory.

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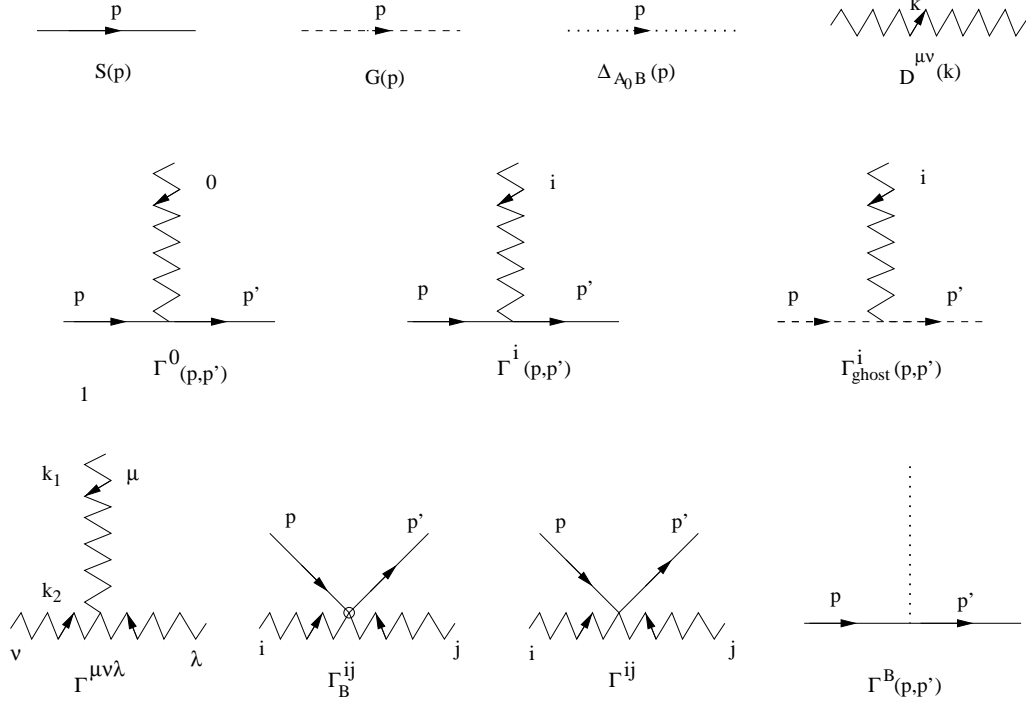


FIG. 1: Feynman rules for the action (1.1).

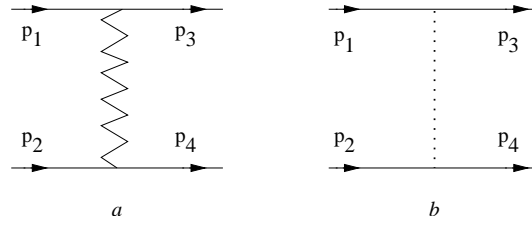


FIG. 2: Tree level scattering.

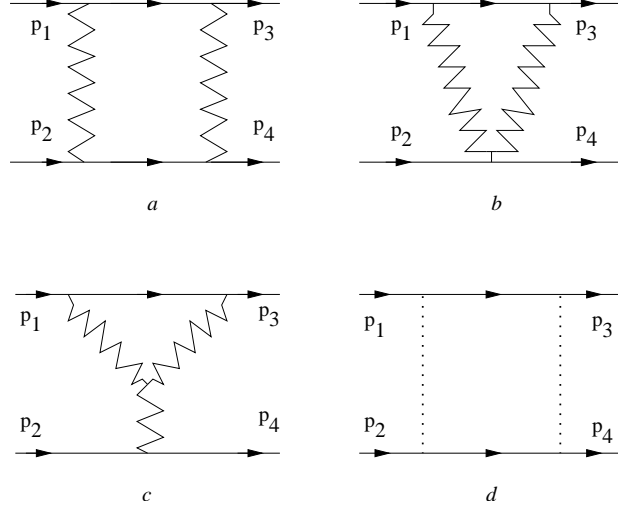


FIG. 3: One-loop scattering.

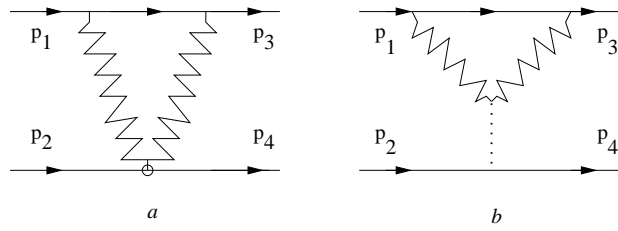


FIG. 4: Pauli's term.